Math 254-1 Exam 6 Solutions

1. Carefully define the Linear Algebra term "independent". Give two examples from $\mathbb{R}^2$.

   A set of vectors is independent if no nondegenerate linear combination yields $\bar{0}$. Any single nonzero vector is independent, such as $\{(1,1)\}$ or $\{(2,3)\}$; also, any basis is independent, such as $\{(1,0),(0,1)\}$.

2. In the vector space $M_{2,3}$ of 2×3 matrices, set $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 0 & 5 \end{pmatrix}, B = \begin{pmatrix} 2 & 4 & 7 \\ 10 & 1 & 13 \end{pmatrix}, C = \begin{pmatrix} 1 & 2 & 5 \\ 8 & 2 & 11 \end{pmatrix}$. Determine whether or not $\{A, B, C\}$ is independent.

   Let E be the standard basis for $M_{2,3}$. Then $[A]_E = [1\ 2\ 3\ 4\ 0\ 5], [B]_E = [2\ 4\ 7\ 10\ 1\ 13], [C]_E = [1\ 2\ 5\ 8\ 2\ 11]$. We put these row matrices into a larger matrix (putting them as columns leads to a different equally valid approach), which we then put into echelon form: $\begin{pmatrix} 1 & 2 & 3 & 4 & 0 & 5 \\ 2 & 4 & 7 & 10 & 1 & 13 \\ 1 & 2 & 5 & 8 & 2 & 11 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 3 & 4 & 0 & 5 \\ 0 & 0 & 1 & 2 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$
   This has two pivots, hence has rank 2, hence $\{A, B, C\}$ is dependent.

3. In the vector space $\mathbb{R}_3[x]$ of polynomials of degree at most 3, set $u_1 = x^3 + x^2 + 2x + 1, u_2 = x^3 - x^2 + x + 1, u_3 = x^3 + 5x^2 + 4x + 1, u_4 = x^3 + 2x^2 + 3x + 4$.

   Set $S = span\{u_1, u_2, u_3, u_4\}$. Find the dimension of S, and a basis.

   Let $E = \{1, x, x^2, x^3\}$ be the standard basis for $\mathbb{R}_3[x]$. We have $[u_1]_E = [\,1\ 2\ 1\ 1\,], [u_2]_E = [\,1\ 1\ -1\ 1\,], [u_3]_E = [\,1\ 4\ 5\ 1\,], [u_4]_E = [\,4\ 3\ 2\ 1\,]$. We put these row matrices into a larger matrix (an alternate solution puts them as columns), which we then put into echelon form: $\begin{pmatrix} 1 & 2 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 4 & 5 & 1 \\ 4 & 3 & 2 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 1 & 1 \\ 0 & -1 & -2 & 0 \\ 0 & 0 & 8 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. This has rank 3; hence $\dim S = 3$. A basis for S is $\{1+x+2x^2+x^3, -x-2x^2, 8x^2-3x^3\}$.

4. In the vector space $\mathbb{R}^2$, set $S = \{(1,1),(4,5)\}$, a basis. Find the change-of basis matrix from the standard basis to S, and use this matrix to find $[(5,-3)]_S$.

   $P_{ES} = ([s_1]_E\ [s_2]_E) = \begin{pmatrix} 1 & 4 \\ 1 & 5 \end{pmatrix}$; $P_{SE} = P_{ES}^{-1} = \begin{pmatrix} 5 & -4 \\ -1 & 1 \end{pmatrix}$ is the desired change-of-basis matrix. We find $[(5,-3)]_S = P_{SE}\begin{pmatrix} 5 \\ -3 \end{pmatrix} = \begin{bmatrix} 37 \\ -8 \end{bmatrix}_S$.

5. In the vector space $\mathbb{R}^3$, set $T = \{(1,1,1),(0,1,2),(1,1,3)\}$, a basis. Find $[(1,2,2)]_T$.

   $P_{ET} = ([t_1]_E\ [t_2]_E\ [t_3]_E) = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix}$; $P_{TE} = P_{ET}^{-1} = \begin{pmatrix} 1/2 & 1 & -1/2 \\ -1 & 1 & 0 \\ 1/2 & -1 & 1/2 \end{pmatrix}$ is the desired change-of-basis matrix, found by applying ERO's to $(P_{ET}|I)$ until we achieve $(I|P_{TE})$. We find $[(1,2,2)]_T = P_{TE}\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} = \begin{bmatrix} 3/2 \\ 1 \\ -1/2 \end{bmatrix}_T$.